Exercises for Stochastic Processes

Tutorial exercises:

T1. Let $S = X^V$, for a topological space X and arbitrary set V. Show that the product topology is the smallest topology on S which makes one-dimensional projections continuous.

T2. Let $S = \{0,1\}^{\mathbb{Z}^d}$. Let $\alpha : \mathbb{Z}^d \to (0,\infty)$ with $\sum_v \alpha(v) < \infty$. The metric

$$\rho(\eta,\xi) := \sum_{v\in\mathbb{Z}^d} \alpha(v)|\eta(v) - \xi(v)|$$

on S generates the topology

$$T_\rho = \{A \subseteq S \ : \ \forall \eta \in A \ \exists r > 0 \text{ such that } B_r(\eta) \subset A\},$$

where

$$B_r(\eta) := \{\xi \in S \ : \ \rho(\eta,\xi) < r\}.$$

Show that T_ρ is equal to the product topology.

T3. Show that a sequence (η_n) in $\{0,1\}^{\mathbb{Z}^d}$ converges w.r.t. the product topology if and only if it converges pointwise.

T4. Show that (with the notations from the lecture)

$$\sum_{x\in V} \sup_\eta |f(\eta_x) - f(\eta)| < \infty$$

does not imply the continuity of a function $f : \{0,1\}^V \to \mathbb{R}$.

→ Page 2

Homework exercises:

H1. Let $\psi(\theta)$ be a characteristic exponent satisfying the Lévy-Khinchin formula and let ξ_t be a random variable with characteristic function $\exp\big(t\psi(\theta)\big)$. Show that

$$T_t f(x) := \mathbb{E}[f(x+\xi_t)]$$

is a probability semigroup.

H2. Let X_t be a Lévy process with Lévy-Khinchin triple (a, σ^2, π). Show that the generator of X_t satisfies

$$\mathcal{L}f(x) = af'(x) + \frac{\sigma^2}{2} f''(x) + \int_{\mathbb{R}} \big(f(x+y) - f(x) - yf'(x)\mathbb{1}_{\{|y|<1\}}\big)\pi(\mathrm{d}y),$$

defined for functions $f \in C_c^\infty(\mathbb{R})$, i.e., infinitely continuously differentiable functions with compact support.

H3. Let X be a compact space and I a countable set. Let $S = X^I$ and associate with this the product topology. Show that S is compact.

Deadline: Monday, 28.01.20